Continuous Probability Distribution

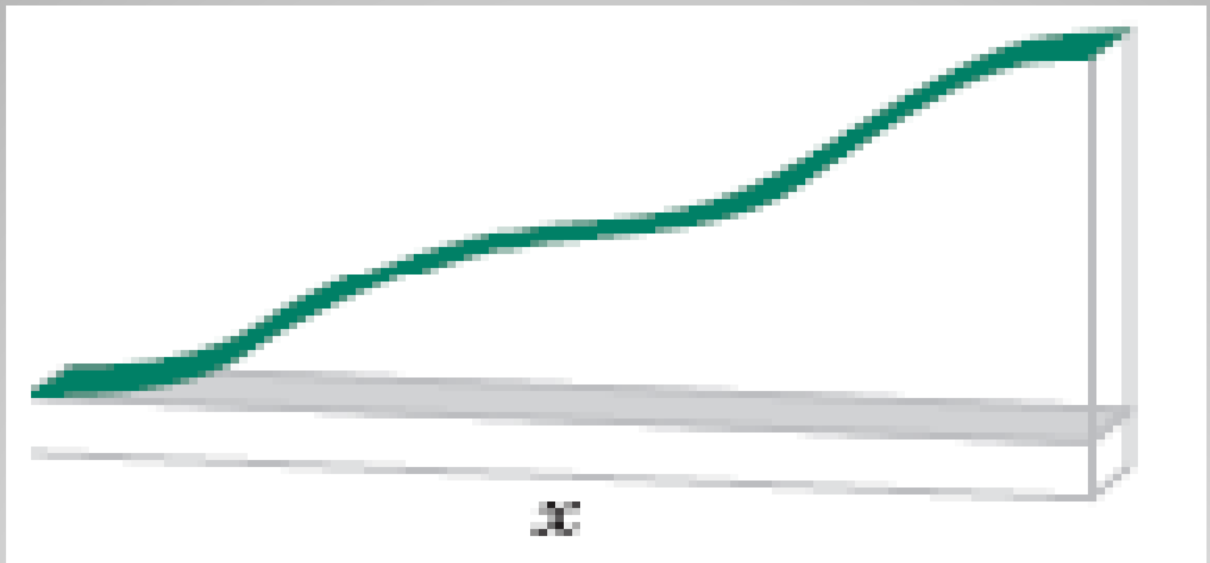
•A continuous random variable X takes all values in an interval of numbers.

Not countable

•The probability distribution of a continuous r.v. *X* is described by a density curve.

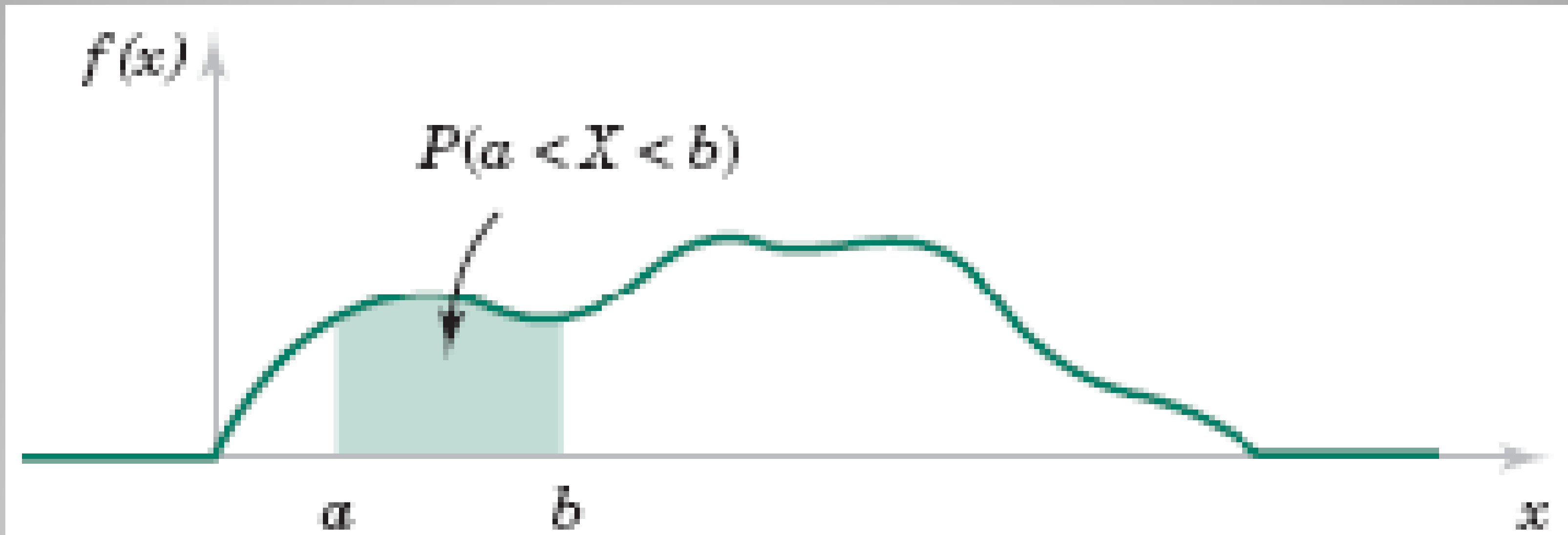
•The probability of any event is the area under the density curve and above the values of X that make up the event.

Continuous Random Variables



Probability Distributions and Probability Density Functions

Figure 3-1 Density function of a loading on a long, thin beam.



Probability Distributions and Probability Density Functions

Figure 3-2 Probability determined from the area under $f(x)$.

Probability Distributions and Probability Density Functions

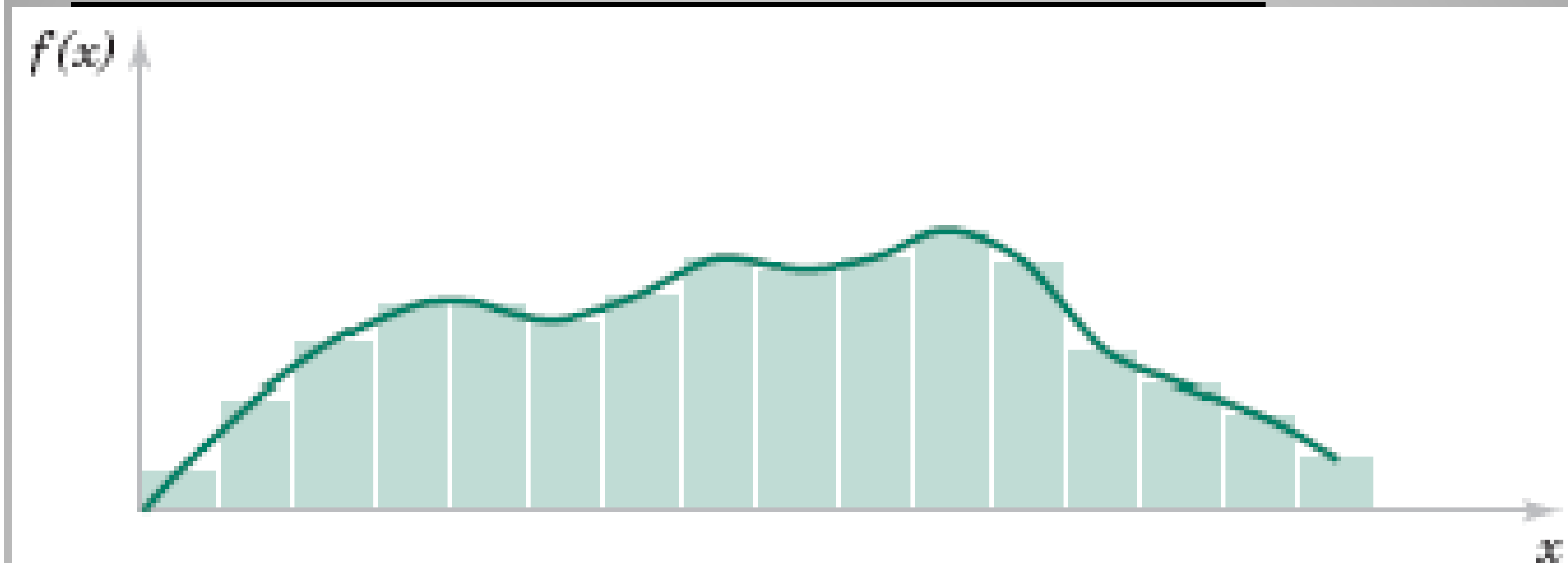
For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx =$ area under $f(x)$ from a to b

For any a and b



Probability Distributions and Probability Density Functions

Figure 3-3 Histogram approximates a probability density function.

If X is a **continuous random variable**, for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (3.2)$$

Probability Distributions and Probability Density Functions

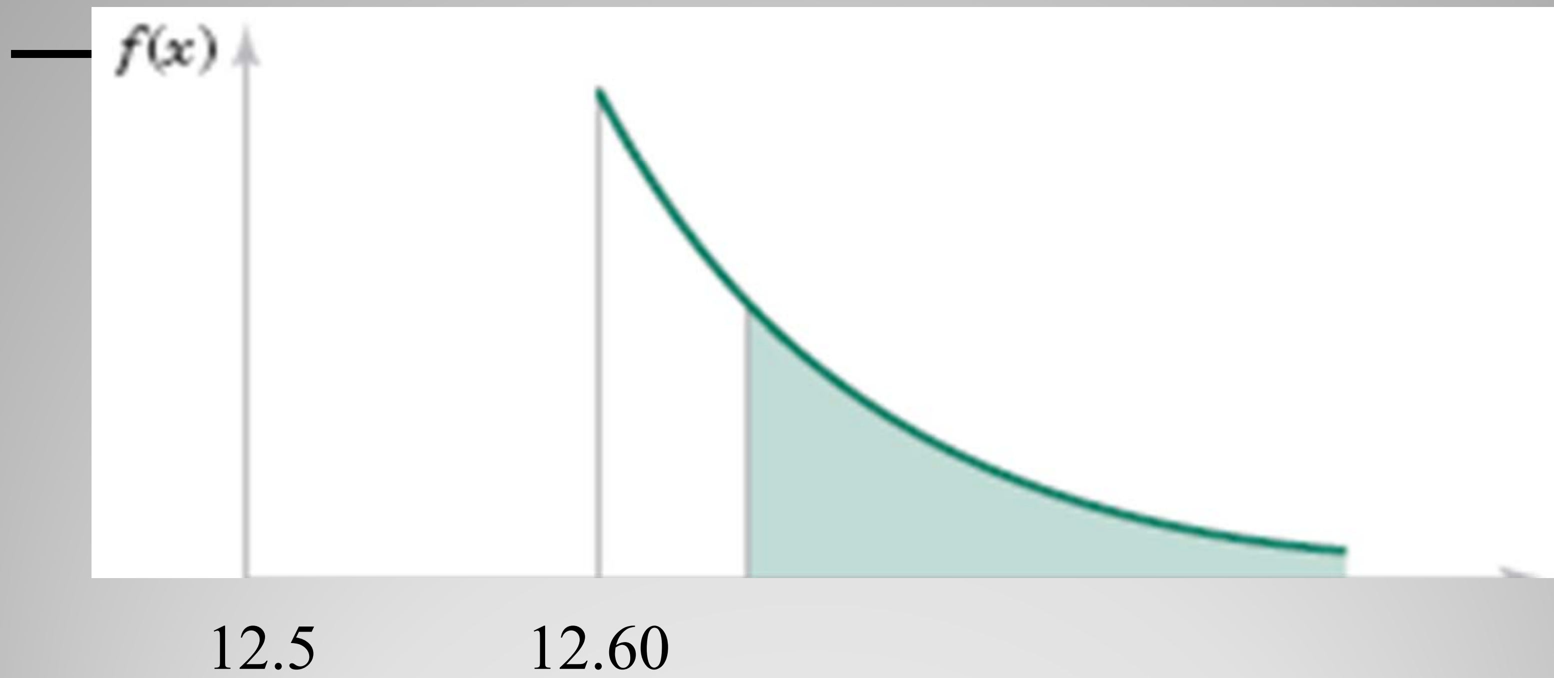
Example 3-2

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scapped? The density function and the requested probability are shown in Fig. 3.5. A part is scrapped if X>12.60. Now,

$$P(X > 12.60) = 20e^{-20(x-12.5)}\,dx = -e^{-20(x-12.5)}\Big|_{12.60}^{\infty} = 0.135$$

Probability Distributions and Probability Density Functions



Probability Distributions and Probability Density Functions

Figure 3.5 Probability density function for Example 3.2.

Example 3-2 (continued)

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\,dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

Probability Distributions and Probability Density Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\, du$$

for $-\infty < x < \infty$.

(3.3)

Cumulative Distribution Functions

Example

For the drilling operation in Example 3.2, F(x) consist of two expressions

$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \leq x$

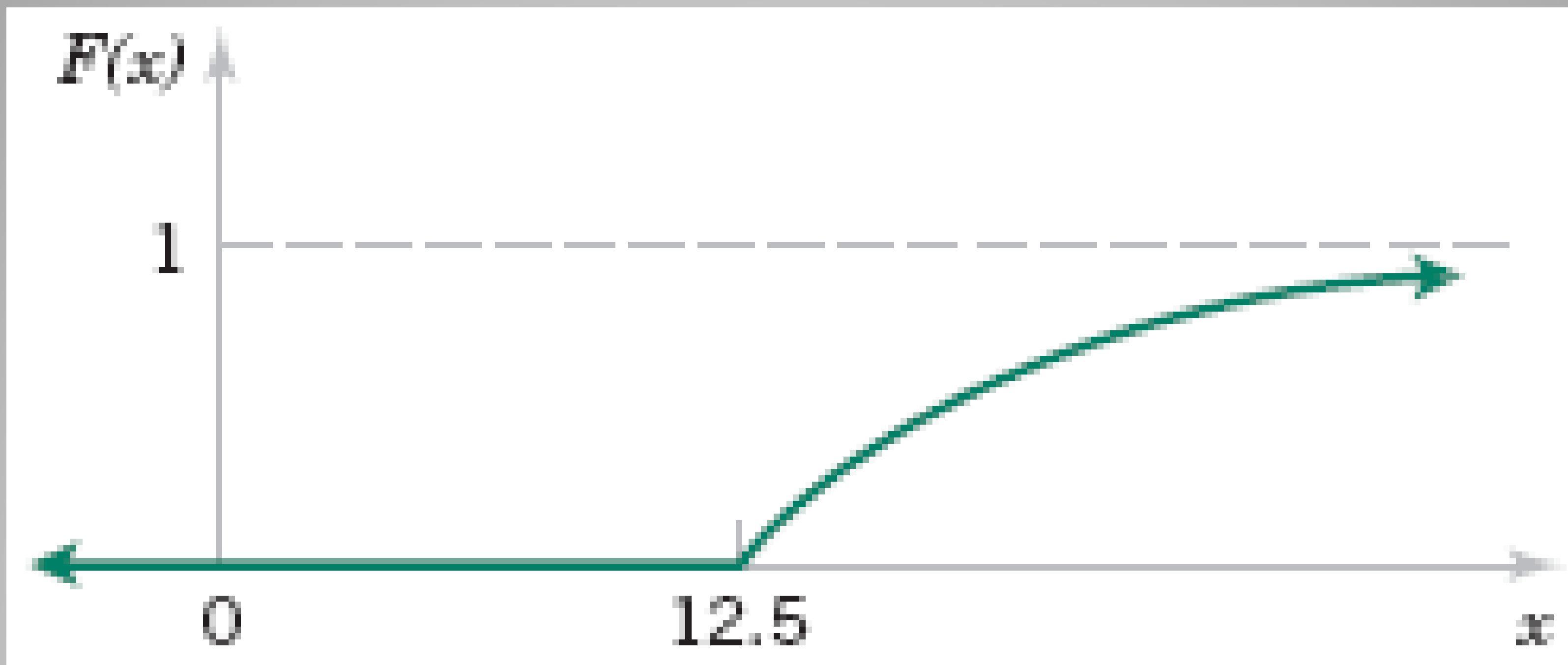
$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\, du$$
$$= 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \leq x \end{cases}$$

Cumulative Distribution Functions

Figure 3.7 display a graph of F(x)



Cumulative Distribution Functions

Figure 3.7 Cumulative distribution function for Example 3.4.

Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function f(x).
The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad (3.4)$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

Mean and Variance of a Continuous Random Variable

Example

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_0^{20} x f(x)\, dx = 0.05x^2/2 \Big|_0^{20} = 10$$

The variance of X is

$$V(X) = \int_0^{20} (x - 10)^2 f(x)\, dx = 0.05(x - 10)^3/3 \Big|_0^{20} = 33.33$$

Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)\,dx \tag{3.5}$$

Mean and Variance of a Continuous Random Variable

Example

For the drilling operation in Example 3.2, the mean of X is

$$E(X) = \int_{12.5}^{\infty} x f(x)\, dx = \int_{12.5}^{\infty} x\, 20 e^{-20(x-12.5)}\, dx$$

Integration by parts can be used to show that

$$E(X) = -x e^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20} \Bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x)\, dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

Continuous Uniform Random Variable

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b \tag{3.6}$$

is a **continuous uniform random variable.**

$$f(x) = \begin{cases} 1/(b\text{-}a), & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Continuous Uniform Random Variable

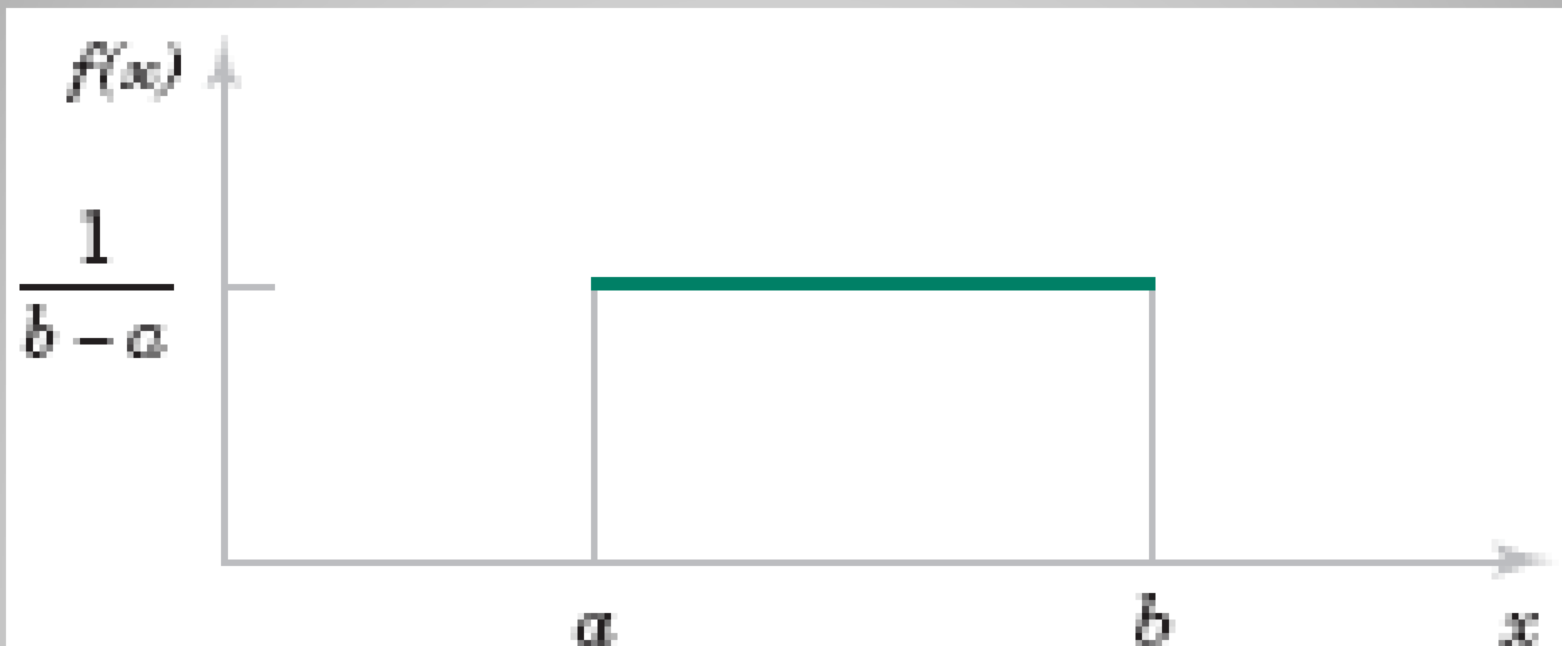


Figure 3.8 Continuous uniform probability density function.

Continuous Uniform Random Variable

Mean and Variance

If X is a continuous uniform random variable over $a \leq x \leq b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12} \tag{3.7}$$

Continuous Uniform Random Variable

Example

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05, 0 \leq x \leq 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

$$P(5 < X < 10) = \int_{5}^{10} f(x)\,dx$$

$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

Continuous Uniform Random Variable

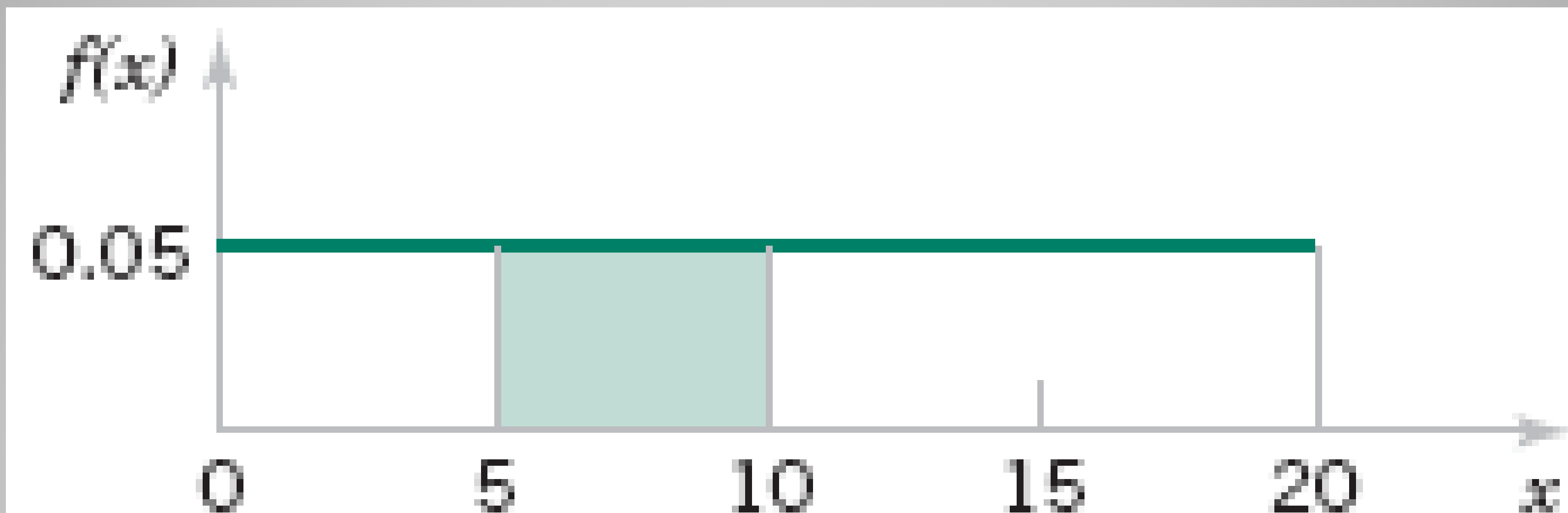


Figure 3.9 Probability for Example 3.9.

Continuous Uniform Random Variable

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

Normal Distribution

Definition

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty \qquad (3.8)$$

Is a normal random variable with parameters μ, where -∞<μ<∞, and σ>0.

Also,

E(X)=μ and V(X)= σ^2 (3.9)

And the notation N(μ, σ^2) is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2, respectively

Normal Distribution

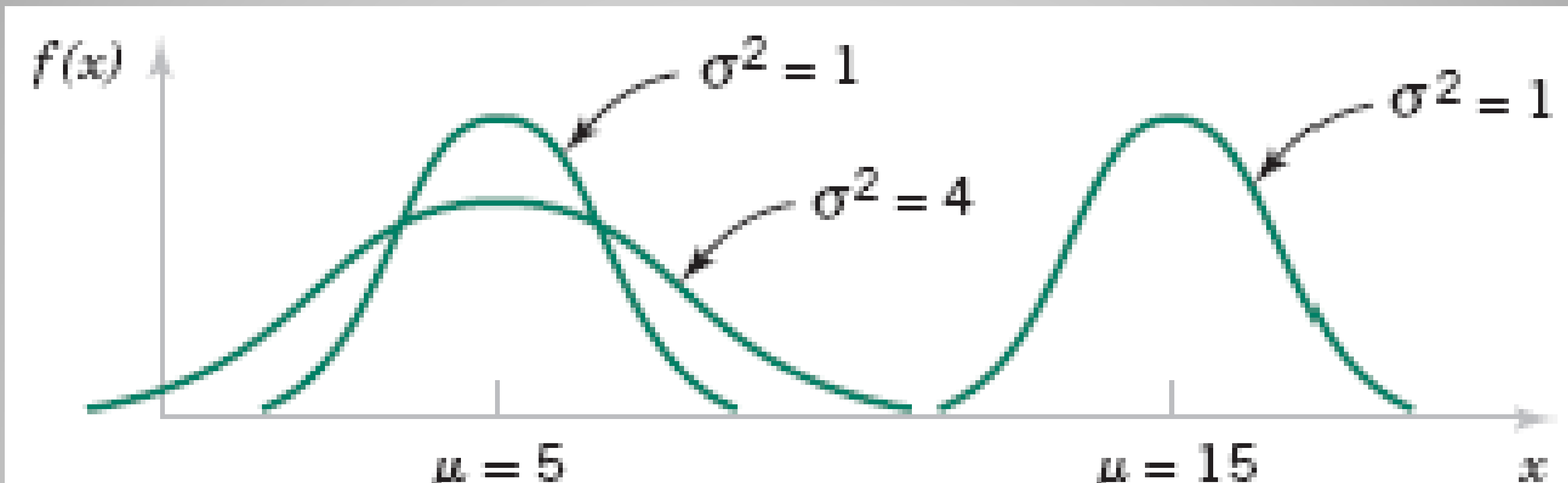


Figure 3.10 Normal probability density functions for selected values of the parameters μ and σ^2.

Normal Distribution

Some useful results concerning the normal distribution

For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

Normal Distribution

Definition : Standard Normal

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z.

The cumulative distribution function of a standard normal random variable is denoted as

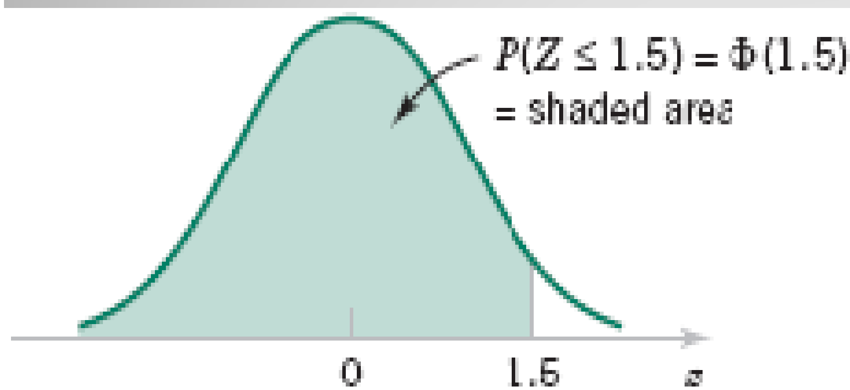
$$\Phi(z) = P(Z \leq z)$$

Normal Distribution

Example

Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form P(Z≤z) . the use of Table II to find P(Z≤1.5) is illustrated in Fig. 3.13. Read down the z column to the row that equals 1.5. the probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column heading refer to the hundredth's digit of the value of z in P(Z≤z). For example, P(Z≤1.53) is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.9399.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 3.13 Standard normal probability density function.

Normal Distribution

Standardizing

If X is a normal random variable with E(X)=μ and V(X)= σ^2 , the random var

$$Z = \frac{X - \mu}{\sigma} \qquad (3.10)$$

Is a normal random variable with E(Z)=0 and V(Z)=1. That is , Z is a standard normal random variable

Normal Distribution

Example

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

Normal Distribution

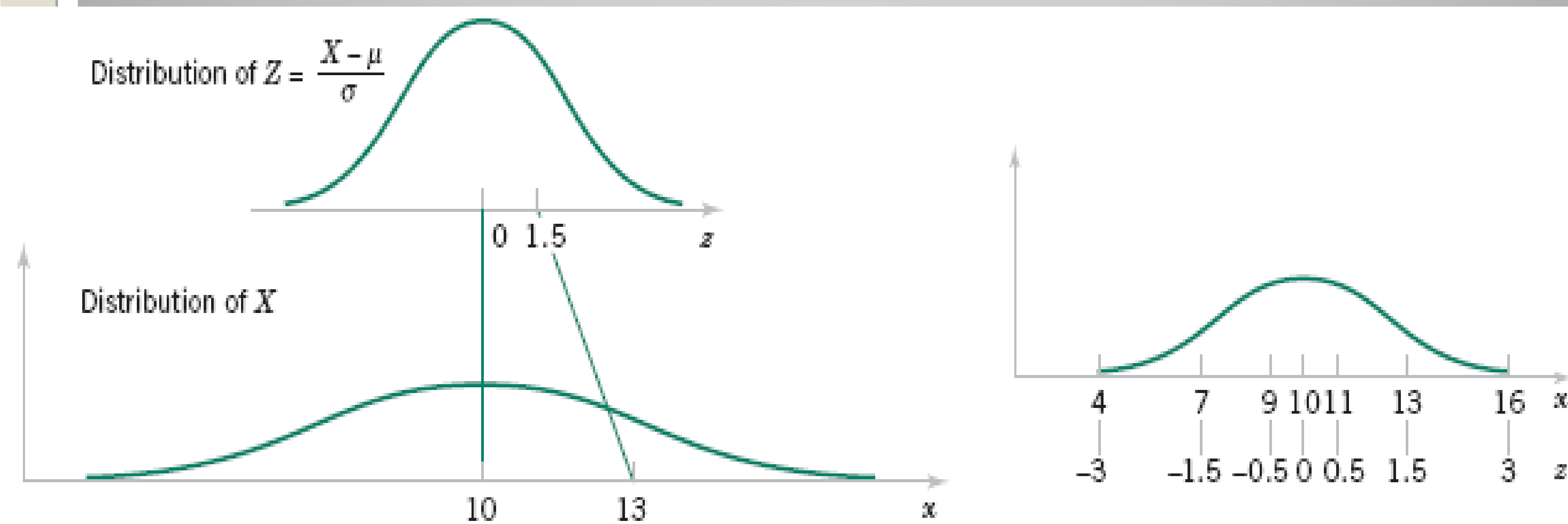


Figure 3.15 Standardizing a normal random variable.

Normal Distribution

To Calculate Probability

Suppose X is a normal random variable mean μ and variance σ^2. Then,

$$P(X \le x) = P\left(\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = P(Z \le z) \qquad (3.11)$$

Where Z is a standard normal random variable, and $z = \frac{(x-\mu)}{\sigma}$

is the z-value ontained by standarding X.

The probability is obtained by entering Appendix Table II with $z = \frac{(x-\mu)}{\sigma}$

Normal Distribution

Example

Continuity the previous example, what is the probability that a current measurement is between milliamperes? From Fig. 3.15, or by proceeding algebraically, we have

$$
\begin{aligned}
P(9 < X < 11) &= P((9-10)/2 < (X-10)/2 < (11-10)/2) \\
&= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\
&= 0.69146 - 0.30854 = 0.38292
\end{aligned}
$$

Normal Distribution

Example (continued)

Determine the value for which the probability that a current measurement is below this value is 0.98. the requested value is shown graphically in Fig. 3.16. we need the value of x

$$
\begin{aligned}
P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\
&= P(Z < (x - 10)/2) \\
&= 0.98
\end{aligned}
$$

Appendix Table II is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x. The result is

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

3.6 Normal Distribution

Example 3.14 (continued)

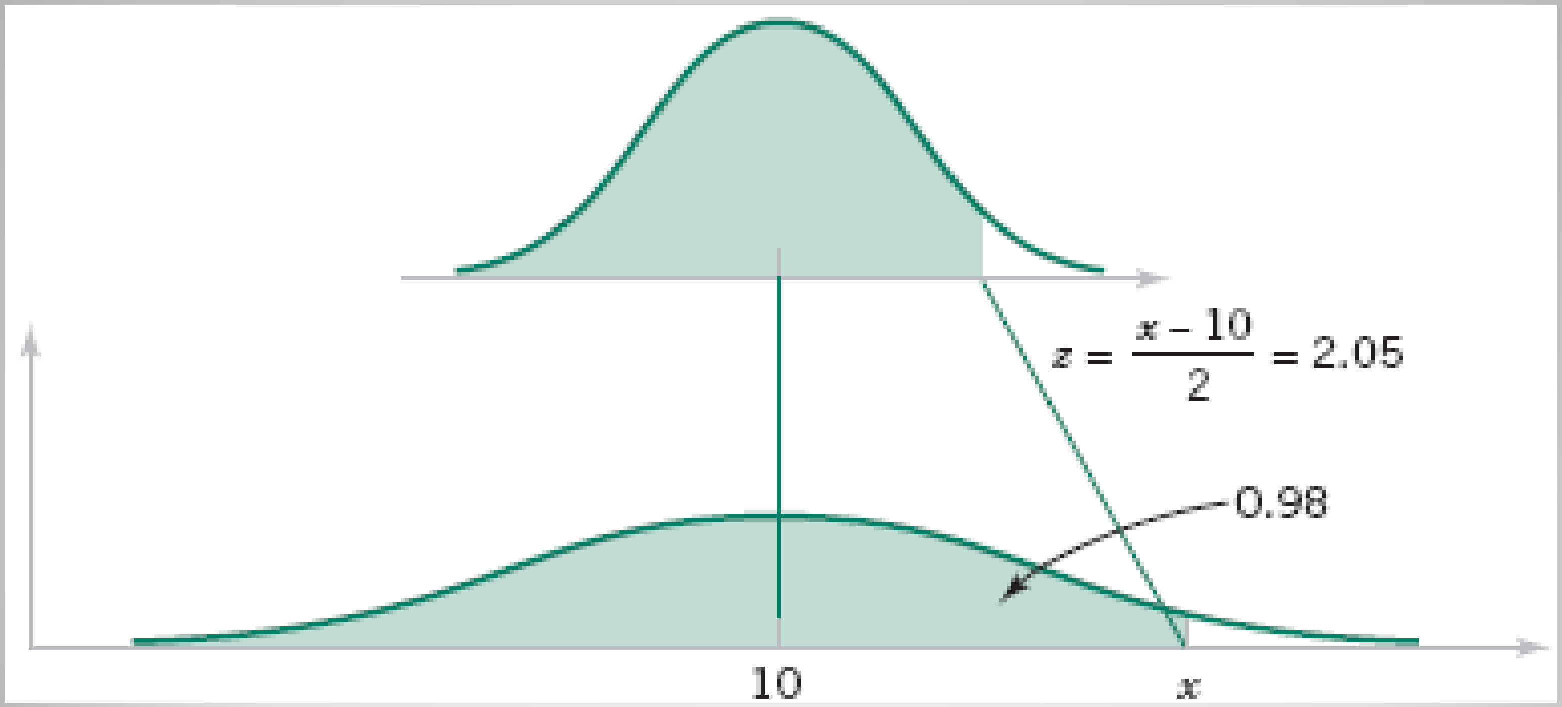


Figure 3.16 Determining the value of *x* to meet a specified probability.

Assignment

1) The contents of Urn I,II and III are as follows 1 white, 2 black and 3 red balls, 2 white, 1 black and 1 red ball, and 4 white , 5 Black and 3 red balls drawn. They happen to be white and red. What is the probability that they come from I,II or III?
2) Compute the variance of the probability distribution of the number of doublets in four throws of a pair of die.
3) The probability that a man aged 50 years will die with in a year is 0.01125. what is the probability that out of 12 such men, at least 11 will reach their fifty first birthday?
4) Fit a Binomial distribution to the following frequency distribution:

X:	0	1	2	3	4	5	6
f:	13	25	52	58	32	16	4

5) Compute the variance of the probability distribution of the number of doublets in four throws of a pair of dice.